ON PERIODIC SOLUTIONS OF A CERTAIN NONLINEAR EQUATION

(O PERIODICHESKOM RESHENII ODNOGO NELINEINOGO URAVNENIIA)

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The question of the existence of periodic solutions of equation [2]

$$\frac{d^2z}{dt^2} - \frac{2e\sin t}{1+e\cos t}\frac{dz}{dt} + \frac{\mu}{1+e\cos t}\sin z = \frac{4e\sin t}{1+e\cos t} \qquad (0 \leqslant e < 1) \qquad (1)$$

is considered by means of Cesari's [1] method.

This equation is invariant with respect to the simultaneous interchange of z and $-z$, and t and $-t$, and hence its solution may supposed to be odd in t. Equation (1) may be rewritten

$$\frac{d^2x}{dt^2} + \frac{e\cos t}{1+e\cos t}x + \mu\sin\frac{x}{1+e\cos t} = 4e\sin t \qquad \left(z = \frac{x}{1+e\cos t}\right) \qquad (2)$$

Consider the space S of functions of integrable square on the interval $[0, 2\pi]$, which are odd and periodic with period 2π, endowed with the norm

$$\nu(x) = \left[\frac{1}{2\pi}\int_0^{2\pi} x^2(t)\,dt\right]^{1/2} \qquad (3)$$

For x in S, we have

$$x \sim \sum_{k=1}^{\infty} b_k \sin kt, \qquad \nu(x) = \left[\frac{1}{2}\sum_{k=1}^{\infty} b_k^2\right]^{1/2} \qquad (4)$$

Consider, in the space S, the linear operator P and H such that

$$Px = b_1\sin t + b_2\sin 2t, \qquad Hx = -\sum_{k=1}^{\infty} k^{-2} b_k \sin kt \qquad (5)$$

If $Px \equiv 0$, then

$$x(t) \sim \sum_{k=3}^{\infty} b_k \sin kt, \qquad \nu(Hx) = \left[\frac{1}{2}\sum_{k=3}^{\infty} k^{-4} b_k^2\right]^{1/2} \leqslant 3^{-2}\nu(x) \tag{6}$$

Now consider the operators Q and F

$$Qx = -\frac{e\cos t}{1+e\cos t}x - \mu\sin\frac{x}{1+e\cos t} + 4e\sin t, \qquad Fx = H(Qx - PQx) \tag{7}$$

Let x^* be a function such that

$$Px^* = x^* = b_1\sin t + b_2\sin 2t \qquad (x^* \in S) \tag{8}$$

Then

$$\nu(x^*) = 2^{-1/2}(b_1^2 + b_2^2)^{1/2} \leqslant c \qquad (c = \text{const}) \tag{9}$$

Consider the subset S^*

$$S^* = [x,\ x \in S,\ Px = x^*,\ \nu(x) \leqslant d,\ \nu(x - x^*) \leqslant \delta] \qquad (d,\ \delta = \text{const}) \tag{10}$$

For arbitrary values of the constants d and δ, the subset S^* is not empty, since each x^* belongs to S^*. The space S is complete, and S^* is closed; consequently, S^* is a complete space.

Following [1], let us introduce the operator T

$$y = Tx = Px + Fx = Px + H(Qx - PQx) \cdot \tag{11}$$

If x is in S^*, then

$$Py = PPx + PFx = x^*, \qquad y - Py = H(Qx - PQx) \tag{12}$$

From this, employing (6), we obtain

$$\nu(y - Py) \leqslant 3^{-2}\nu(Qx - PQx) \tag{13}$$

Employing the obvious inequality

$$\nu(z - Pz) \leqslant \nu(z), \qquad z \in S$$

one obtains readily that

$$\nu(Qx - PQx) \leqslant \nu\left[\frac{e\cos t}{1+e\cos t}Px - P\left(\frac{e\cos t}{1+\cos t}Px\right)\right] + \\ + \nu\left[\frac{e\cos t}{1+e\cos t}(x - Px)\right] + \mu\nu\left[\sin\frac{x}{1+e\cos t}\right] \leqslant \left[F(b_1, b_2, e) - \right. \\ \left. - \frac{\gamma_1^2(b_1, b_2, e) + \gamma_2^2(b_1, b_2, e)}{2}\right]^{1/2} + \frac{e\delta}{1-e} + \mu \tag{14}$$

Here

$$F(b_1, b_2, e) = \frac{1}{2\pi}\int_0^{2\pi}\Phi^2(t)\,dt \qquad \left(\Phi(t) = \frac{e\cos t}{(1+e\cos t)}x^*(t)\right)$$

where γ_1 and γ_2 are the coefficients of sin t and of sin $2t$ in the Fourier expansion of the function $\Phi(t)$.

Now suppose that the coefficients b_1 and b_2 of equation (8) satisfy the inequalities

$$|b_1| \leqslant \alpha_1, \qquad |b_2| \leqslant \alpha_2 \qquad (\alpha_1, \alpha_2 = \text{const}) \tag{15}$$

Then we may put

$$c = 2^{-1/2} (\alpha_1^2 + \alpha_2^2)^{1/2} \tag{16}$$

From (14) we obtain that

$$\nu (Qx - PQx) \leqslant \max\nolimits_D \left[F - \frac{\gamma_1^2 + \gamma_2^2}{2}\right]^{1/2} + \frac{e\delta}{1-e} + \mu \tag{17}$$

where D is the closed rectangle in the b_1, b_2 space which is defined by (15).

In order that y belong to S^*, it is obviously sufficient that

$$\nu (y - Py) \leqslant \delta, \qquad d = c + \delta \tag{18}$$

The first of relations (18) will be fulfilled, provided that

$$\frac{1}{3^2} \left\{\max\nolimits_D \left[F - \frac{\gamma_1^2 + \gamma_2^2}{2}\right]^{1/2} + \frac{e\delta}{1-e} + \mu\right\} \leqslant \delta \tag{19}$$

If (19) holds for a certain $\delta = \delta_0$, then, upon setting

$$d = c + \delta_0 \tag{20}$$

we obtain

$$T (S^*) \in S^* \tag{21}$$

Let us prove that, in a certain range of values of the parameters μ and e, the operator T is a contracting operator on S^*. Writing $y_1 = Tx_1$ $y_2 = Tx_2$, with x_1 and x_2 in S^*, we obtain

$$y_1 - y_2 = H\,[(Qx_1 - Qx_2) - P\,(Qx_1 - Qx_2)]$$

$$\nu (y_1 - y_2) \leqslant 3^{-2}\nu\,[(Qx_1 - Qx_2) - P\,(Qx_1 - Qx_2)] \leqslant 3^{-2}\,\nu\,(Qx_1 - Qx_2)$$

$$\nu (y_1 - y_2) \leqslant \frac{1}{3^2}\,\nu \left[\frac{e\cos t}{1 + e\cos t}(x_1 - x_2) + \mu\left(\sin\frac{x_1}{1 + e\cos t} - \sin\frac{x_2}{1 + e\cos t}\right)\right] \leqslant$$
$$\leqslant \frac{1}{3^2}\left[\frac{e}{1-e}\,\nu\,(x_1 - x_2) + \frac{\mu}{1-e}\,\nu\,(x_1 - x_2)\right] \leqslant \frac{1}{3^2}\,\frac{\mu + e}{1-e}$$

Hence, if

$$\frac{1}{3^2}\,\frac{\mu + e}{1-e} < 1 \tag{22}$$

then T is a contracting operator on S^*.

Thus, if relations (19) to (22) are fulfilled, then the operator T, in view of Banach's contraction mapping theorem [3], has a unique fixed point y in S^*. It may be proved [1], that y is a continuous function of x^*, and, therefore, of the coefficients b_1 and b_2. We have

$$y = Py + H(Qy - PQy) \tag{23}$$

The right-hand side of equation (23) is, almost everywhere, a twice continuously differentiable function of t, in view of hypothesis H; consequently, the left-hand side also has this property.

Differentiating (23) twice, we obtain

$$\frac{d^2y}{dt^2} = P\left(\frac{d^2y}{dt^2}\right) + Qy - PQy = Qy + P\left[\frac{d^2y}{dt^2} - Qy\right] \tag{24}$$

The function $y(t)$ is a periodic solution of (2), when $P[d^2y/dt^2 - Qy] \equiv 0$ or, which is the same thing, when

$$U = -b_1 + \beta_1(b_1, b_2) - 4e = 0, \qquad V = -4b_2 + \beta_2(b_1, b_2) = 0 \tag{25}$$

$$\beta_n = \frac{1}{\pi}\int_0^{2\pi} \frac{e\cos t}{1+e\cos t}\, y(t)\sin nt\, dt + \frac{\mu}{\pi}\int_0^{2\pi} \frac{y(t)}{1+e\cos t}\sin nt\, dt \qquad (n = 1,2)$$

Equations (25) may be regarded as equations for the determination of the coefficients b_1 and b_2. The question of the existence of a periodic solution of (2) reduces, therefore, to the question of the existence of solutions b_1 and b_2 of equations (25) which also satisfy the inequalities (15). Let us substitute equations (25) by the approximate equations

$$U_0 = -b_1 + \beta_{10}(b_1, b_2) - 4e = 0, \qquad V_0 = -4b_2 + \beta_{20}(b_1, b_2) = 0 \tag{26}$$

$$\beta_{n0} = \frac{1}{\pi}\int_0^{2\pi} \frac{e\cos t}{1+e\cos t}\, x^*(t)\sin nt\, dt + \frac{\mu}{\pi}\int_0^{2\pi} \sin\frac{x^*(t)}{1+e\cos t}\sin nt\, dt \qquad (n = 1, 2)$$

Let us map, by means of formulas (26), the domain D of the b_1, b_2 plane into a domain in the UV plane.

Let c_0 be the boundary of the corresponding domain Δ_0 in the UV plane. If the origin of coordinates of the UV plane belongs to Δ_0, then system (26) has a solution satisfying (15). If, in particular, the following inequality holds:

$$\max_D \sqrt{(U - U_0)^2 + (V - V_0)^2} < \min\,[c_0, 0] \tag{27}$$

where min $[c_0, 0]$ is the least distance from the origin of coordinates to a point of the boundary c_0, then the domain Δ, obtained as a result of the mapping of D by means of formulas (25), also contains the origin (0, 0), and hence (25) also has a solution which satisfies (15).

Now consider the question of the existence of periodic solutions of (2) for $0 \leqslant \mu \leqslant 1$ and $e = 0.6$. Condition (22) holds for these values of the parameters. Let us choose $\alpha_1 = 3.5$ and $\alpha_2 = 0.5$, then inequality (19) has the form

$$3^{-2}\,[0.415 + 1.5\delta + \mu] \leqslant \delta \tag{28}$$

and is fulfilled for $\delta = 0.056 + 0.134\,\mu$. Equation (20) then gives $d = 2.556 + 0.134\,\mu$. Let us estimate the quantity appearing on the left-hand side of (27)

$$|U - U_0| = |\beta_1 - \beta_{10}| = \left| \frac{1}{\pi}\int_0^{2\pi} \frac{e\cos t}{1+\cos t}(y - Py)\sin t\,dt + \right.$$

$$\left. + \frac{\mu}{\pi}\int_0^{2\pi} 2\sin\frac{y - Py}{2(1+e\cos t)}\cos\frac{y+Py}{2(1+e\cos t)}\sin t\,dt \right|$$

Employing the Cauchy-Buniakovskii inequality for integrals, we obtain

$$|U - U_0| \leqslant \frac{1}{\pi}\left(\int_0^{2\pi}(y - Py^2)\,dt\right)^{1/2}\left(\int_0^{2\pi}\frac{e^2\sin^2 t\cos^2 t}{(1+c\cos t)^2}\,dt\right)^{1/2} +$$

$$+ \frac{\mu}{\pi}\left(\int_0^{2\pi}(y - Py)^2\,dt\right)^{1/2}\left(\int_0^{2\pi}\frac{\sin^2 t}{(1+e\cos t)^2}\,dt\right)^{1/2} \leqslant$$

$$\leqslant \left(\frac{2}{\pi}\right)^{1/2}\delta\left[\left(\int_0^{2\pi}\frac{e^2\sin^2 t\cos^2 t}{(1+e\cos t)^2}\,dt\right)^{1/2} + \mu\left(\int_0^{2\pi}\frac{\sin^2 t}{(1+e\cos t)^2}\,dt\right)^{1/2}\right] \tag{29}$$

Analogously, it follows that

$$|V - V_0| \leqslant \left(\frac{2}{\pi}\right)^{1/2}\delta\left[\left(\int_0^{2\pi}\frac{e^2\sin^2 2t\cos^2 t}{(1+e\cos t)^2}\,dt\right)^{1/2} + \mu\left(\int_0^{2\pi}\frac{\sin^2 2t}{(1+e\cos t)^2}\,dt\right)^{1/2}\right] \tag{30}$$

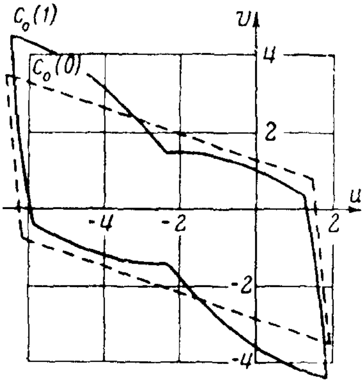
Choosing $c = 0.6$, and calculating the integrals appearing in (29) and (30), we obtain

$$\max_t \sqrt{(U - U_0)^2 + (V - V_0)^2} \leqslant \delta\sqrt{6.473\,\mu^2 + 5.299\,\mu + 1.103} \tag{31}$$

For $\mu = 1$, from (31) it follows that

$$\max_D \sqrt{(U - U_0)^2 + (V - V_0)^2} < 0.682 \tag{32}$$

Let us map the rectangle $|b_1| \leqslant 3.5$, $|b_2| \leqslant 0.5$, of the b_1, b_2 plane, into the UV plane, by means of formulas (26), for $\mu = 1$. Then we obtain the domain (see Figure), bounded by the curve c_0 (1), where



$$\min\,[c_0(1), 0] > 0.780 \qquad (33)$$

From (32) and (33) it follows that inequality (27) holds.

If we map the same rectangle into the UV plane, again using (26) but with $\mu = 0$, we obtain the domain (see Figure) which is bounded by the curve c_0 (0).

From the Figure it is readily seen that

$$\min\,[c_0(1), 0,] < \min\,[c_0(0), 0] \qquad (34)$$

The quantities U_0, V_0 are linear functions of μ; hence, for $0 \leqslant \mu \leqslant 1$, the following inequality holds:

$$\min\,[c_0(1), 0] \leqslant \min\,[c_0(\mu), 0] \qquad (35)$$

On the other hand, the quantity appearing on the right-hand side of (31) is an increasing function of μ for $0 \leqslant \mu \leqslant 1$, and thus inequality (32) is also true for all μ in the interval $[0,\ 1]$. From this it follows easily that (27) holds for $e = 0.6$ and $0 \leqslant \mu \leqslant 1$. For these values of the parameters, system (27) has solutions b_1, b_2 satisfying (15), and hence equation (2) has periodic solutions.

The mapping of the boundary of the rectangle into the UV plane was carried out by means of the computing machine "Strela".

BIBLIOGRAPHY

1. Cesari, L., Periodic solutions of differential systems. *Proc. Symposium on Active Networks and Feedback Systems*. New York, 1960; Brooklyn, N.Y., 1961.

2. Beletskii, V.V., O libratsii sputnika (On librations of satellites). *Iskusstvennye sputniki Zemli*, No. 3, 1959.

3. Kantorovich, L.V. and Akilov, G.P., *Funktsional'nyi analiz v normirovannykh prostranstvakh (Functional Analysis in Normed Spaces)*. Fizmatgiz, 1959.

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